UDC 539.3

EXACT SOLUTION OF PLANE PROBLEMS ON THE CONTACT BETWEEN SEMI-INFINITE BEAMS AND AN ELASTIC WEDGE

PMM Vol. 39, № 6, 1975, pp. 1100-1109
G. Ia. POPOV and L. Ia. TIKHONENKO
(Odessa)
(Received July 1, 1974)

The solution of the following problem is given.

1. Identical semi-infinite beams with stiffness D make contact with both faces of a wedge $(0 \leqslant r < \infty, -\alpha \leqslant \theta \leqslant \alpha)$. The loads applied normally to beams $q_1(r)$ and $q_2(r)$ can be distinct. The ends of the beams which coincide with the edges of the wedge are interrelated differently. The problem is solved by partitioning into a symmetric (1a) and antisymmetric (1b) problems. Problem 1a can be treated as a problem on the bending of a beam lying on a wedge $(0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \alpha)$, whose second face is under sliding support conditions.

2. A semi-infinite, normally loaded beam is impressed in one of the faces of a wedge $(0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \alpha)$ while the other face is either free (Problem 2a) or rigidly fixed (Problem 2b).

3. Both sides of a semi-infinite normally loaded beam make contact with wedges $(0 \leqslant r < \infty, -\beta \leqslant \theta \leqslant 0)$ and $(0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \alpha)$, whose materials can be different.

The exact solution of the problems listed is obtained by the method stated in [1]. This method is perfected and simplified here, which permits substantially to expand the range of problems solved as compared with [1] where only some particular cases of Problem 2a were considered. On the basis of the exact solutions obtained, the nature of the singularity in the contact stress σ_θ near the point of the wedge is investigated for all the problems listed and for the whole range of the angle α. It is found that Problem 1b is not correct for angles $\alpha \geqslant {}^1/_2\,\pi$.

The investigation is carried out without taking account of the shear contact stress, without taking account of the phenomenon of beam separation from the wedge, and for the case of plane strain in the wedge.

1. Problem 1. In the symmetric case (Problem 1a), both beams are loaded by the compressive load ${}^1/_2\,[q_1(r) + q_2(r)]$, while in the anti-symmetric case (Problem 1b) one beam $(\theta = \alpha)$ is loaded by the compressive load ${}^1/_2\,[q_1(r) - q_2(r)]$ and the other $(\theta = -\alpha)$ is loaded by the same load but with a separating effect. For simplification of the notation, $q(r)$ will denote the load on the beam everywhere below.

Assuming the contact stresses positive, we obtain the boundary conditions of Problem 1a as

$$\tau_{r\theta} = 0, \quad D\frac{\partial^4 v}{\partial r^4} = \mp\sigma_\theta \mp q(r), \quad \theta = \pm\alpha \tag{1.1}$$

The boundary conditions of Problem 1b differ from (1.1) only in that a minus is retained in front of $q(r)$, also for $\theta = -\alpha$.

Still another condition on the connection of the beam ends to the point of the wedge must be added to conditions (1.1). Three kinds of connections are possible: a rigid connection, a hinge connection and a free edge. The equilibrium conditions of the system of two semi-infinite beams show that we can limit ourselves to the examination of the first two connections in the case of Problem 1a since a hinge connection is equivalent to free edges. In the case of Problem 1b all three methods of connecting the beam ends are equivalent to the following equilibrium conditions for one of the beams:

$$\int_0^\infty [\sigma_\theta(r, \alpha) + q(r)] r^k dr = 0, \quad k = 0, 1 \tag{1.2}$$

These conditions should be used in Problem 1a also for the case of hinged connection of the beams. In the rigid connection case, condition (1.2) must be replaced for $k = 1$ by the following:

$$\partial v / \partial r\, (r, \alpha) = 0, \quad r = 0 \tag{1.3}$$

Let us proceed to construct the solution of Problem 1a. Proceeding exactly as in [1], taking account of the symmetry of the problem and realizing the boundary conditions (1.1), we arrive at the relationship

$$\frac{1}{2\pi i} \int_\Omega \lambda \prod_{n=1}^{3} (p+n) F(p) r^{-p-4} dp = \frac{1}{2\pi i} \int_\Omega T(p) F(p) r^{-p-1} dp + \frac{1}{2} q(r) \tag{1.4}$$

$$\lambda = \frac{D(\varkappa + 1)}{4G}, \quad F(p) = \frac{pB(p)}{\csc \alpha\, (p+1)}, \quad T(p) = \frac{1}{2} \left[\frac{p-1}{\operatorname{tg} \alpha\, (p+1)} - \frac{p+1}{\operatorname{tg} \alpha\, (p-1)} \right]$$

Here the contour of integration Ω is the line $\operatorname{Re} p = c$, where $c_0 < c < 0$, and the meaning of the symbols G, $\varkappa$, $B(p)$ and c_0 is the same as in [1].

According to the scheme presented in this paper, the relationship obtained should be reduced to a Carleman problem for a strip. This reduction can be accomplished by two methods: either by shifting the contour of integration to the left in the first integral of (1.4), as has been done in [1], or by shifting the contour of integration to the right in the second integral. Since the methods mentioned are not equivalent in the sense of the constraints imposed on the function $q(r)$, it is expedient to elucidate both.

Let us start with the second method. We use the notation

$$\Phi_1(p) = T(p) F(p), \quad Q_1(p) = \int_0^\infty q(r) r^{p+3} dr \tag{1.5}$$

and we assume that the function $\Phi_1(p)$ is analytic in the strip Π_0 (the notation $\{c + 3m < \operatorname{Re} p < c + 3(m+1)\} = \Pi_m$, $m = 0, \pm 1, \pm 2, \ldots$ is used here and below) and is continuous in the closed strip Π_0. Moreover, uniformly relative to $c \leqslant \sigma \leqslant c + 3$

$$\int_{-\infty}^{\infty} |\Phi_1(\sigma + it)|^2 dt < \text{const} \tag{1.6}$$

As will be shown, the function $\Phi_1(p)$ constructed below actually possesses the properties listed under the condition that $r^{7/4+c} q(r) \in L_2(0, \infty)$ while $Q_1(p) \in H_\Omega$, i.e. satisfies the Hölder condition locally on the line Ω.

These properties of the function $\Phi_1(p)$ permit shifting the contour of integration by three to the right in the right integral of (1.4) by using the Cauchy theorem, which results in a Carleman boundary value problem for a strip.

$$\lambda T^{-1}(p_0)\prod_{n=1}^{3}(p_0+n)\Phi_1(p_0)=\Phi_1(p_0+3)+\frac{1}{2}Q_1(p_0),\quad p_0\in\Omega \quad (1.7)$$

To realize the second method, we use the notation

$$\Phi_2(p)=\prod_{n=1}^{3}(p+n)F(p),\quad Q(p)=\int_0^\infty q(r)r^p dr \quad (1.8)$$

and assume that the function $\Phi_2(p)$ possesses the same properties in the strip Π_{-1} as does the function $\Phi_1(p)$ in Π_0. We show below that it actually possesses these properties under the condition that $r^{1/2+k+c}q^{(k)}(r)\in L_2(0,\infty)$, $k=0, 1, 2, 3,$ while $Q(p)\in H_\Omega$.

Shifting the contour of integration by three to the left in the left integral of (1.4), we arrive at the Carleman problem

$$\lambda\Phi_2(p_0-3)=\prod_{n=1}^{3}(p_0+n)^{-1}T(p_0)\Phi_2(p_0)+\frac{1}{2}Q(p_0),\quad p_0\in\Omega \quad (1.9)$$

The exact solution of the problems (1.7), (1.9) can be obtained, as in [1 — 3], by reducing it [4] to a Riemann problem [5] for which the structure of the exact solution depends principally on the index. As is seen from [1 — 3], finding the index of the Riemann problem is difficult because the coefficient has a complex singularity at the ends of the contour. It should be noted that similar difficulties occur in solving Wiener-Hopf equations of the first kind [6] reduced to the Riemann problem on the axis with a coefficient which also has a singularity. They are there surmounted by using partial factorization. We apply this idea to the Carleman problems (1.7), (1.9) by trying to convert them so that the coefficients would have no singularities and the increment of their arguments would be zero (a similar idea has been used earlier in [7]).

By using the function

$$\Phi(p)=[\lambda^{1/3 p}\Gamma(p+1)\sin {}^1/_2\pi p]^{-1}\Phi_1(p) \quad (1.10)$$

the boundary condition (1.7) is rewritten as

$$K(p_0)\Phi(p_0)+\Phi(p_0+3)=H(p_0),\quad p_0\in\Omega \quad (1.11)$$

$$K(p)=\operatorname{tg}{}^1/_2\pi p T^{-1}(p),\quad H(p)=Q_1(p)[2\lambda^{1/3 p+1}\Gamma(p+4)\cdot\cos{}^1/_2\pi p]^{-1}$$

Now the required function $\Phi(p)$ has simple poles at the points $p_1=0$ and $p_2=2$ in the strip Π_0, but the coefficient of the problem $K(p)$ is continuous on each line Ω lying in the strip $\gamma<\operatorname{Re}p<0$ ($\gamma=\max\{-1, 1-\pi/\alpha\}$), has no zeros, possesses the asymptotics $K(p)=1+O(e^{-2\beta|p|})$, $|p|\to\infty$ ($\beta=\min\{\alpha, {}^1/_2\pi\}$), satisfies the Hölder condition, and finally $[\arg K(p)]_\Omega=0$. All the properties listed are verified sufficiently simply with the exception of the last one. To prove it, we use the asymptotics $T(p)=\pm i$, $\operatorname{Im}p\to\pm\infty$, which permits finding that $\Delta=[\arg T(p)]_\Omega=n\pi$, where n is an integer, odd, and independent of α. Setting $\alpha={}^1/_2\pi$ in the expression for $T(p)$, we find by analogy with [1] (see Sect. 4) that $\Delta=-\pi$ from which follows what is required.

Using the function

$$\omega(w)=\frac{1}{\sqrt{-w}}\Phi\left(\frac{3i\ln(-w)}{2\pi}+c\right)$$

we pass from the Carleman problem (1.11) to the Riemann problem [1, 4]

$$K_1(u)\omega^+(u) = \omega^-(u) + h(u), \quad u<0 \tag{1.12}$$

$$K_1(u) = K\left(\frac{3i\ln(-u)}{2\pi} + c\right), \quad h(u) = \frac{1}{\sqrt{-u}} H\left(\frac{3i\ln(-u)}{2\pi} + c\right)$$

The properties noted for the functions $\Phi(p)$, $K(p)$ and $Q_1(p)$ permit obtaining the solution of problem (1.12) by the scheme presented in [5]. Then, returning from $\omega(w)$ to the function $\Phi(p)$, we obtain formulas yielding the solution of the Carleman problem (1.11)

$$\Phi(p) = X(p)\left[\frac{1}{6i}\int_\Omega \frac{H(s)\,ds}{X(s+3)\sin{}^1\!/_3\pi(p-s)} + \frac{C_1}{\sin{}^1\!/_3\pi p} + \frac{C_2}{\sin{}^1\!/_3\pi(p-2)}\right] \tag{1.13}$$

$$X(p) = \exp\left\{\frac{1}{3}\int_\Omega [e^{2/_3\pi i(s-p)} - 1]^{-1}\ln K(s)\,ds\right\}, \quad p\in\Pi_0$$

Here C_1 and C_2 are arbitrary constants.

Let us examine the properties of the constructed function $\Phi(p)$. Analyzing (1.13), we note that for any integer m the function $\Phi(p)$ is analytic in each strip Π_m, with the exception of the points $p_1 = 3m$ and $p_2 = 3m + 2$, where there are simple poles, and it has a jump on each line $\{\mathrm{Re}\, p = c + 3m\} = \Omega_m$, where its limit values to the left $(\Phi_-(p))$ and right $(\Phi_+(p))$ on this line are connected by the relationship

$$\Phi_-(p) = K(p-3m)\Phi_+(p) - (-1)^m H(p-3m), \quad p\in\Omega_m \tag{1.14}$$

$$\Phi_+(p) = X_+(p)\left[\frac{1}{2}\frac{H(p)}{X(p+3)} + \frac{1}{6i}\int_\Omega \frac{H(s)\,ds}{X(s+3)\sin{}^1\!/_3\pi(p-s)} + \frac{C_1}{\sin{}^1\!/_3\pi p} + \frac{C_2}{\sin{}^1\!/_3\pi(p-2)}\right]$$

$$X_+(p) = \exp\left\{-\frac{1}{2}\ln K(p) + \frac{1}{3}\int_\Omega [e^{2/_3\pi i(s-p)} - 1]^{-1}\ln K(s)\,ds\right\}, \ p\in\Omega$$

Taking into account that the functions $K(p)$ and $H(p) \in H_\Omega$, it can be shown [1] that $\Phi(p)$ is continuous in each closed strip Π_m, with the exception of the points p_1 and p_2, and therefore, the function $\Phi_1(p)$ is continuous and analytic in the closed strip Π_0

There remains to show compliance with condition (1.6) for the constraint imposed above on the load. According to [8], this constraint is equivalent to the condition $Q_1(p) \in L_2(\Omega)$. From (1.11) and the asymptotics for the gamma function known from [9], it follows that $H(p)$ and $p^{7/_3+c}H(p) \in L_2(\Omega)$. Using the Fourier transform (after the substitution $s = c + i\tau$, $p = c + iz$) and Theorem 68 from [8], it can be shown that the integral

$$\frac{1}{6i}\int_\Omega [X(s+3)\sin{}^1\!/_3\pi(p-s)]^{-1} H(s)\,ds, \quad p\in\Pi_0$$

possesses this same property.

Subsequent use of (1.10) and (1.13) and the asymptotics for the gamma function permits making the deduction that the function $\Phi_1(p)$ satisfies condition (1.6).

In the case of a hinged connection between the beams, the constants C_1 and C_2 are determined by conditions (1.2), which it is convenient to use in the form

$$\Sigma\ (p,\ \alpha) + Q\ (p) = 0, \quad p = 0,\ 1 \tag{1.15}$$

Here $\Sigma\ (p,\ \alpha)$ denotes the Mellin transform of the contact stress $\sigma_\theta\ (r,\ \alpha)$. Noting that

$$\Sigma\ (p,\ \alpha) = 2[\Phi_1\ (p)]_+, \quad Q\ (p) = Q_1(p-3)$$

the conditions (1.15) are realized. Consequently, we obtain equations to determine these constants

$$\Phi\ (1) + Q_1\ (-2)(2\ \sqrt[3]{\lambda})^{-1} = 0, \quad C_1 = -Q_1\ (-3)[3\ X(0)]^{-1} \tag{1.16}$$

In the case of a rigid connection between the beams, the realization of condition (1.3) is complex. In order to simplify it, we multiply the beam bending equation by r^2 and integrate (by parts) from zero to infinity. This transformation results in the expression

$$\int_0^\infty [\sigma_\theta\ (r,\ \alpha) + q\ (r)]\ r^2 dr = 2D\ \frac{\partial v}{\partial r}\ (r,\ \alpha)\,|_{r=0} = 0$$

Therefore, condition (1.3) is equivalent to the relationship $\Sigma\ (2,\ \alpha) + Q\ (2) = 0$. Realization of this latter and of condition (1.15) for $p = 0$ yields

$$C_1 = -Q_1\ (-3)[3X\ (0)]^{-1}, \quad C_2 = Q_1(-1)[6\lambda^{2/3}X\ (2)]^{-1}$$

Thus, the exact solution of Problem 1a has been obtained. This permits writing all the quantities of interest in quadratures. For example, the contact stress $\sigma_\theta\ (r,\ \alpha)$ has the form

$$\sigma_\theta\ (r,\ \alpha) = \frac{1}{2\pi i} \int_\Omega 2\lambda^{1/3\, p}\, \Gamma\ (p+1) \sin \frac{1}{2}\pi p\ \Phi_+\ (p) r^{-p-1} dp \tag{1.17}$$

Here the function $\Phi_+\ (p)$ is defined by (1.14).

Now, investigating the behavior of the stress (1.17) at the wedge apex and at infinity according to the scheme in [1], we find that the asymptotics for $r \to 0$ is valid in the case of hinge connected beams: $\sigma_\theta = O\ (\ln r)$ for $\alpha \leqslant {}^1/_2\pi$ and $\sigma_\theta = O\ (r^{\pi/\alpha - 2})$ for ${}^1/_2\ \pi < \alpha < \pi$, while $\sigma_\theta = O\ (1)$ and $\sigma_\theta = O\ (r^{\pi/\alpha - 2})$, respectively, in the case of a rigid connection. As $r \to \infty$ the contact stress σ_θ decreases as $r^{-\delta}$ independently of the method of beam connection, $\delta = \min\ (4,\ \varepsilon)$, where ε enters into the asymptotics $q\ (r) = O\ (r^{-\varepsilon})$, $r \to \infty$. The results obtained for $\alpha = {}^1/_2\ \pi$ agree with the results in [10].

Applying the method of partial factorization to the problem (1.9), we arrive at the following problem:

$$\Phi\ (p_0 - 3) + K\ (p_0)\ \Phi\ (p_0) = H\ (p_0), \quad p_0 \in \Omega \tag{1.18}$$

$$\Phi\ (p) = [\lambda^{1/3\, p} \cos {}^1/_2 \pi p\ \Gamma\ (p+4)]^{-1} \Phi_2\ (p),\ K\ (p) = \operatorname{ctg} {}^1/_2\ \pi p T\ (p)$$

$$H\ (p) = -[2\lambda^{1/3 p}\ \Gamma\ (p+1) \sin {}^1/_2 \pi p]^{-1} Q\ (p)$$

We note that the function $\Phi\ (p)$ has two simple poles $p_1 = -3$ and $p_2 = -1$, in the strip Π_{-1}, the coefficient of the problem (1.18) possesses the same properties as the coefficient of the problem (1.11) and the free term (under the constraint made above on the function $q\ (r)$) possesses the properties: $H\ (p) \in H_\Omega$, $p^{1/2+k+c} H\ (p) \in L_2\ (\Omega)$, $k = 0,\ 1,\ 2,\ 3$. The solution of the problem (1.18) is constructed by the same method as the problem (1.11) and has the form

$$\Phi(p) = X^{-1}(p)\left[\frac{1}{6i}\int\limits_{\Omega}\frac{X(s-3)H(s)\,ds}{\sin{}^1/_3\pi(s-p)} + \frac{C_1}{\sin{}^1/_3\pi p} + \frac{C_2}{\sin{}^1/_3\pi(p+1)}\right] \quad (1.19)$$

The function $X(p)$ is here determined by the second of formulas (1.13).

Performing an analysis similar to the analysis of (1.13), we arrive at the deduction that the function $\Phi_2(p)$ in the strip Π_{-1} possesses all the properties assumed earlier. Interchanging the plus and minus subscripts of $\Phi(p)$ in (1.14) yields a formula connecting the limit values of the function (1.19) on the line Ω_m.

Let us present the expression for the contact stress $\sigma_\theta(r, \alpha)$

$$\sigma_\theta(r,\alpha) = \frac{-1}{2\pi i}\int\limits_{\Omega}\frac{(\sin 2\alpha p + p\sin 2\alpha)\,\Gamma(p+1)\,\Phi_-(p)\,\lambda^{1/_3 p}}{\sin\alpha(p-1)\sin\alpha(p+1)\,\mathrm{sc}^1/_2\pi p}\,r^{-p-1}dp \quad (1.20)$$

In the case of a hinged connection between the beams, the constants C_1 and C_2 are determined by the formula

$$C_1 = -\frac{\sqrt{3}}{12i}\int\limits_{\Omega} X(s-3)H(s)\csc\frac{1}{3}\pi(s-1)\,ds, \quad C_2 = 0$$

while in the case of a rigid connection $C_1 = C_2 = 0$.

Investigation of the integral (1.20) shows that the stress asymptotics $\sigma_\theta(r, \alpha)$ agrees completely with the corresponding asymptotics of the stress (1.17) as $r \to 0$ for both kinds of connections (just as for $r \to \infty$).

Thus, on the basis of the Carleman problems (1.7) and (1.9), two forms of the solution of problem 1a have been obtained. For a rigorous foundation of the solution obtained on the basis of problem (1.9), it is hence required that $r^{1/_2+k+c}q^{(k)}(r) \in L_2(0, \infty)$, $k = 0, 1, 2, 3$, and on the basis of (1.7) that $r^{7/_2+c}q(r) \in L_2(0, \infty)$. Therefore, the selection of the form of the solution of the contact problem should be linked up with a preliminary analysis of the properties of the function $q(r)$.

We obtain the solution of the remaining problems on the basis of the Carleman problem of the form (1.7). Preference is given to this form of the solution because it includes the case of a piecewise-constant load $q(r)$ which is of the greatest practical interest (it is possible to pass from this case over to the case of a concentrated force).

Let us briefly examine Problem 1b. Taking its anti-symmetry into account and realizing the boundary conditions, we arrive at the Carleman problem (1.7) for which the functions $T(p)$ and $\Phi_1(p)$ are determined by the formulas

$$T(p) = {}^1/_2[(p+1)\,\mathrm{tg}\,\alpha(p-1) - (p-1)\,\mathrm{tg}\,\alpha(p+1)] \quad (1.21)$$
$$\Phi_1(p) = T(p)\,p\cos\alpha(p+1)\,B(p)$$

It is easy to see that the function

$$K(p) = \mathrm{tg}\,{}^1/_2\,\pi p\,T^{-1}(p) \quad (1.22)$$

possesses the same properties in the strip $\gamma < \mathrm{Re}\,p < 0$, $\gamma = \max\{-1, 1-\pi/2\alpha\}$ as the coefficient of the problem (1.11).

Using the function (1.10), we pass from the problem (1.7), (1.21) to the problem (1.11), (1.21), (1.22), whose solution is determined by (1.13). Independently of the kind of connection of the beam ends, the constants C_1 and C_2 are determined by (1.16), and the integral (1.17) yields an expression for the contact stress σ_θ. Analyzing the behavior of the stress at the wedge apex, we find that for $\alpha < {}^1/_6\,\pi$ the quantity $\sigma_\theta(r, \alpha)$

tends to zero as r when $r \to 0$, as $r \ln r$ for $\alpha = {}^1/_6\,\pi$, as $r^{\pi/2\alpha-2}$ for ${}^1/_6\,\pi < \alpha \leqslant {}^1/_4\,\pi$ and has a power-law singularity of the form $r^{\pi/2\alpha-2}$ for ${}^1/_4\,\pi < \alpha < {}^1/_2\pi$. For $\alpha \geqslant {}^1/_2\,\pi$ the singularity becomes nonintegrable, therefore, Problem 1b is correct only for convex angles. The contact stress decreases at infinity exactly as in the previous problem.

2. Problem 2a. A beam lies freely on a face $(\theta = 0)$ of a wedge $(0 \leqslant r < \infty,\ 0 \leqslant \theta \leqslant \alpha < 2\pi)$. Then the boundary conditions for the wedge and the equilibrium conditions for the beam are

$$\sigma_\theta, \tau_{r\theta}(r, \alpha) = 0; \quad \tau_{r\theta} = 0, \quad D\frac{\partial^4 v}{\partial r^4} = \sigma_\theta + q(r), \quad \theta = 0 \tag{2.1}$$

$$\int_0^\infty [\sigma_\theta(r, 0) + q(r)]\, r^k dr = 0, \quad k = 0, 1$$

Realization of the boundary conditions (2.1) results in the Carleman problem (1.7), in which we should set

$$T(p) = 2(\sin^2\alpha p - p^2 \sin^2\alpha)(\sin 2\alpha p + p \sin 2\alpha)^{-1} \tag{2.2}$$

$$\Phi_1(p) = T(p)\, p\, B(p)$$

and which reduces to the problem (1.11) by means of (1.10). Evidently the function

$$K(p) = {}^1/_2 \operatorname{tg} {}^1/_2\pi p\, (\sin 2\alpha p + p \sin 2\alpha)(\sin^2\alpha p - p^2 \sin^2\alpha)^{-1} \tag{2.3}$$

possesses all the necessary properties in the strip $\gamma < \operatorname{Re} p < 0$, where $\gamma = \max\{-1, c_1\}$, and c_1 is the first negative root of the equation $\sin 2\alpha p + p \sin 2\alpha = 0$. Since the line Ω is in this strip, i.e. $\gamma < \operatorname{Re} p_0 < 0$, then the solution of the problem (1.11), (2.3) is given by (1.13). As before, the constants C_1 and C_2 are determined by (1.16).

Particular cases of Problem 2a (a load in the form of concentrated forces P and a moment M applied to the end of the beam coincident with the point of the wedge, and $\alpha = {}^1/_2\pi, \pi, {}^3/_2\pi$) were examined in [1]. It is clear that consideration of this load case is carried out by the usual scheme. The function $\Phi(p)$ is hence determined from (1.13) in which is set

$$H(p) \equiv 0, \quad C_2 = C_1 + \sqrt{3}\, M\, [4\sqrt[3]{\lambda}\, X(1)]^{-1}, \quad C_1 = -P\,[3X(0)]^{-1}$$

For any kind of load, (1.17) yields the expression for the contact stress $\sigma_\theta(r, \alpha)$. An analysis of the behavior of this stress as $r \to 0$ shows that for $\alpha < {}^1/_2\pi$ the quantity σ_θ tends to zero as r^μ, $\mu = \max\{1, -c_1 - 1\}$, it is bounded for $\alpha = {}^1/_2\pi$ but has a power singularity of the form r^{-c_1-1} for $\alpha > {}^1/_2\pi$. The behavior of σ_θ as $r \to \infty$ is the same as in Problem 1. These results agree with analogous results in [1, 11].

Problem 2b. In this case the boundary conditions are

$$u, v = 0, \quad \theta = \alpha; \quad \tau_{r\theta} = 0, \quad D\frac{\partial^4 v}{\partial r^4} = \sigma_\theta + q(r), \quad \theta = 0 \tag{2.4}$$

One of the following conditions for fixing the end of the beam

$$\frac{\partial v}{\partial r}(r, 0) = 0, \quad r = 0; \quad \int_0^\infty [\sigma_\theta(r, 0) + q(r)]\, r dr = 0 \tag{2.5}$$

should be added.

The first condition characterizes rigid, and the second hinge fixing (the free edge condition is equivalent to hinge fixing).

The realization of conditions (2.4) results in a Carleman problem (1.7) for which

$$\Phi_1(p) = T(p)\, pB(p) \tag{2.6}$$

$$T(p) = \tfrac{1}{2}[4\varkappa \sin^2 \alpha p + 4p^2 \sin^2\alpha - (\varkappa + 1)^2](\varkappa \sin 2\alpha p - p \sin 2\alpha)^{-1}$$

Let c_2 and c_3 be the greatest real parts of the roots of the equations $4\varkappa \sin^2\alpha p + 4p^2 \sin^2\alpha - (\varkappa + 1)^2 = 0$ and $\varkappa \sin 2\alpha p - p \sin 2\alpha = 0$ in the left half-plane, respectively. Then the function $T(p)$ is continuous in the strip $\gamma < \operatorname{Re} p < 0$, $\gamma = \max\{-1, c_2, c_3\}$, does not vanish, and has the asymptotics $T(p) = \pm i$, $\operatorname{Im} p \to \pm\infty$. Moreover, $[\arg T(p)]_\Omega = \pi$. Hence, the factor $\tan \tfrac{1}{2}\pi p$ which improves the coefficient of problem (1.7) does not appear. The factor $-\cot \tfrac{1}{2}\pi p$ replaces it here. In fact, the function

$$K(p) = -2 \operatorname{ctg}\tfrac{1}{2}\pi p\, (\varkappa \sin 2\alpha p - p \sin 2\alpha)[4\varkappa \sin^2 \alpha p + 4p^2 \sin^2\alpha - (\varkappa + 1)^2]^{-1} \tag{2.7}$$

possesses all the necessary properties in the strip $\gamma < \operatorname{Re} p < 0$. By using the function (1.10), in which the $\sin \tfrac{1}{2}\pi p$ is replaced by $\cos \tfrac{1}{2}\pi p$, we go over from the problem (1.7), (2.6) to the problem (1.11), (2.7). Hence, $\cos \tfrac{1}{2}\pi p$ in the expression for $H(p)$ should be replaced by $-\sin \tfrac{1}{2}\pi p$. Since the function $\Phi(p)$ now has only one pole, the point $p = 1$ in the strip Π_0, then the first formula in (1.13) is changed and becomes

$$\Phi(p) = X(p)\left[\frac{1}{6i}\int_\Omega \frac{H(s)\, ds}{X(s+3)\sin \frac{1}{2}\pi(p-s)} + \frac{C}{\sin \frac{1}{2}\pi(p-1)}\right]$$

The realization of conditions (2.5) yields equations for the constant C

$$C = Q_1(-2)[3\sqrt[3]{\lambda}\, X(1)]^{-1}, \quad \Phi(2) = \tfrac{1}{4}\lambda^{-4/3} Q_1(-1)$$

The first equation corresponds to the case of a free end of a beam, and the second one, to rigid fixing.

The contact stress $\sigma_\theta(r, 0)$ is determined by the integral (1.17) in which the $\sin \tfrac{1}{2}\pi p$ must be replaced by $\cos \tfrac{1}{2}\pi p$. Analyzing this integral, we detect that independently of the method of fixing the end of the beam, the quantity σ_θ is bounded for $\alpha < \tfrac{1}{2}\pi$ as $r \to 0$, has a logarithmic singularity for $\alpha = \tfrac{1}{2}\pi$, and the singularity becomes a power in the form r^{-c_3-1} for $\alpha > \tfrac{1}{2}\pi$.

P r o b l e m 3, Let the beam make contact on one of its sides with the face $\theta = 0$ of a wedge A $(0 \leqslant r < \infty, -\beta \leqslant \theta \leqslant 0)$, and on the other with the face $\theta = 0$ of a wedge B $(0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \alpha)$. Let the superscript 1 mark all quantities characterizing the stress state of the wedge A and the superscript 2, of the wedge B. Then the boundary conditions and the condition of a free edge for the beam are written as

$$\sigma_\theta^{(1)} = \tau_{r\theta}^{(1)} = 0, \quad \theta = -\beta; \qquad \sigma_\theta^{(2)} = \tau_{r\theta}^{(2)} = 0, \quad \theta = \alpha \tag{2.8}$$

$$\tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)} = 0, \quad v^{(1)} = v^{(2)}, \quad \theta = 0$$

$$D\frac{\partial^4 v^{(2)}}{\partial r^4} = \sigma_\theta^{(2)} - \sigma_\theta^{(1)} + q(r), \quad \theta = 0$$

$$\int_0^\infty [\sigma_\theta^{(2)} - \sigma_\theta^{(1)} + q(r)]\, r^k dr = 0, \quad k = 0, 1, \quad \theta = 0$$

Realizing the boundary conditions written down, we arrive at the Carleman problem (1.7). Hence

$$T(p) = \frac{D}{4\lambda}\left[\frac{2G^{(2)}}{\varkappa^{(2)}+1}\frac{\sin^2\alpha p - p^2\sin^2\alpha}{\sin 2\alpha p + p\sin 2\alpha} + \frac{2G^{(1)}}{\varkappa^{(1)}+1}\frac{\sin^2\beta p - p^2\sin^2\beta}{\sin 2\beta p + p\sin 2\beta}\right] \qquad (2.9)$$

$$\lambda = \frac{D}{4}\left[\frac{G^{(2)}}{\varkappa^{(2)}+1} + \frac{G^{(1)}}{\varkappa^{(1)}+1}\right]^{-1}, \quad \Phi_1(p) = \frac{4\lambda(\varkappa^{(2)}+1)}{DG^{(2)}}\, T(p)\, pB^{(2)}(p)$$

We assume that the function $T(p)$ has no zeros in the strip $\gamma < \operatorname{Re} p < 0$, $\gamma = \max\{-1, c_1^{(1)}, c_1^{(2)}\}$, where $c_1^{(1)}$ and $c_1^{(2)}$ are the greatest real parts of the roots of the equations $\sin 2\beta p + p\sin 2\beta = 0$ and $\sin 2\alpha p + p \sin 2\alpha = 0$ in the left half-plane, respectively.

This is verified sufficiently simply for $\alpha = \beta$. Moreover, the following considerations might also be a confirmation. In the case of the presence of zeros for the function $T(p)$ in the strip $\gamma < \operatorname{Re} p < 0$ it can be shown by relying on the principle of the argument that the solution of problem (1.7), (2.9) will contain a number of arbitrary constants, not equal to two, and this will result either in a contradiction to the uniqueness of the solution of the mechanical problem being examined, or in the nonexistence of its solution.

Taking this hypothesis as a basis, it is easy to show that the function $T(p) \in H_\Omega$ if $\gamma < \operatorname{Re} p_0 < 0$ and that $[\arg T(p)]_\Omega = -\pi$. Then the function (1.10) reduces the problem (1.7), (2.9) to the problem (1.11), (2.9), whose solution is given by (1.13). The arbitrary constants C_1 and C_2 in the solution are determined by the equilibrium conditions (2.8) for the beam. It is easy to go from these conditions over to (1.16) which determine the constants C_1 and C_2, using the formulas

$$\Sigma^{(2)}(p,0) - \Sigma^{(1)}(p,0) + Q(p) = 0, \quad p = 0, 1$$

$$2[\Phi_1(p)]_+ = \Sigma^{(2)}(p,0) - \Sigma^{(1)}(p,0)$$

The expressions for the contact stresses $\sigma_\theta^{(1)}(r,0)$ and $\sigma_\theta^{(2)}(r,0)$ are obtained from the integral (1.17) by multiplying the integrand, respectively, by the functions $f^{(1)}(p)$ and $f^{(2)}(p)$ of the form

$$f^{(1)}(p) = -DG^{(1)}(\sin^2\beta p - p^2\sin^2\beta)[2\lambda(\varkappa^{(1)}+1)(\sin 2\beta p + p\sin 2\beta)T(p)]^{-1}$$

$$f^{(2)}(p) = DG^{(2)}(\sin^2\alpha p - p^2\sin^2\alpha)[2\lambda(\varkappa^{(2)}+1)(\sin 2\alpha p + p\sin 2\alpha)T(p)]^{-1}$$

An analysis of the behavior of the contact stresses as $r \to 0$ shows that for $\beta, \alpha < {}^1/_2\pi$ the stresses $\sigma_\theta^{(1,2)}(r,0)$ have a zero of the order $\mu^{(1,2)} = \max\{1, c_1^{(1,2)} - 1\}$, these stresses are bounded for $\beta = \alpha = {}^1/_2\pi$ and have a power singularity of the order $-c_1^{(1,2)} - 1$ for $\beta, \alpha > {}^1/_2\pi$, respectively.

The method by which the exact solution of Problem 3 has been obtained permits obtaining exact solutions for a whole series of problems which differ from Problem 3 only by the boundary conditions on the faces $\theta = -\beta$ and $\theta = \alpha$ (including inhomogeneous conditions).

REFERENCES

1. Popov, G. Ia. and Tikhonenko, L. Ia., Two-dimensional problem of the contact between a semi-infinite beam and an elastic wedge. PMM Vol. 38, № 2, 1974.
2. Tikhonenko, L. Ia., Plane mixed problem of heat conduction for a wedge. Differentsial'nye Uravneniia, Vol. 9, № 10, 1973.
3. Tikhonenko, L. Ia., Plane contact problem for an elastic wedge and a coupled semi-infinite elastic rod. In: Stability and Strength of Structural Elements, Dnepropetrovsk Univ. Press, 1973.
4. Cherskii, Iu. I., Normally solvable smooth transition equations. Dokl. Akad. Nauk SSSR, Vol. 190, № 1, 1970.
5. Gakhov, F. D., Boundary Value Problems. (English translation), Pergamon Press, Book № 10067, 1966 (Distributed in the USA by the Addison-Wesley Publ. Co.).
6. Popov, G. Ia., On an integro-differential equation. Ukr. Matem. Zh., Vol. 12, № 1, 1960.
7. Bantsuri, R. D., Contact problem for a wedge with elastic support. Dokl. Akad. Nauk SSSR, Vol. 211, № 4, 1973.
8. Titchmarsh, E., Introduction to Fourier Integral Theory. Gostekhizdat, Moscow-Leningrad, 1948.
9. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series and Products. Fizmatgiz, Moscow, 1963.
10. Popov, G. Ia., On the analysis of an infinite hinge-slit beam slab on an elastic half-space. Izv. VUZ, Stroit. i Arkhit., №3, 1959.
11. Popov, G. Ia., Bending of a semi-infinite plate resting on a linearly deformable foundation. PMM Vol. 25, № 2, 1961.

Translated by M. D. F.

UDC 539.3

ON THE STABILITY OF THE NATURAL UNSTRESSED STATE OF VISCOELASTIC BODIES

PMM Vol. 39, № 6, 1975, pp. 1110-1117

L. P. LEBEDEV
(Rostov-on-Don)

(Received January 2, 1975)

Within the framework of the Cauchy problem, a class of models of a linear viscoelastic body subjected to the stability principle of the natural unstressed state state of viscoelastic bodies (Principle Y) is isolated in [1]. The principle Y is formulated as follows. Let the boundary conditions be such that the appropriate elasticity theory problem has a zero solution. If a viscoelastic body is free of external loads at each instant $t > 0$, then for every initial state, strain of the body vanishes as $t \to \infty$. The principle Y is called partial if it is satisfied only for some particular class of viscoelasticity problems.

Sufficient conditions for compliance with the partial Y principle are obtained in this paper for models of viscoelastic bodies within the framework of the fun-